Balance Laws in Dynamic Fracture

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ABSTRACT: This work aims at the study of the dynamic fracture of an elastic material in the framework of the configurational mechanics. The analysis is based on the global balances for the physical and configurational fields. Thus, the concept of the balance law for an elastic fractured body, in Euclidean and material space, is treated in detail. In the spirit of modern continuum mechanics, a rigorous localization process is proposed. This procedure provides the equations in Euclidean and material space as well as the new contributions for the configurational forces and moments at the crack tip. In addition, it facilitates the derivation of the relationship between the energy release rate (or the rotational release rate) and the configurational force (or the configurational moment). The results are compared with the corresponding ones of fracture mechanics and some new interpretations are discussed.

1 INTRODUCTION

The propagation of a crack of any curvature in a deformable body is a complex phenomenon because apart from the dynamics of the elastic motion, the evolution of the crack must be accounted for, too. The evolution of the crack takes place, not in the physical space, but within the material body, that is, the material space. Thus, we believe that configurational mechanics (Maugin 1993, 1995; Gurtin 2000) should be the appropriate framework in which this problem can be efficiently studied.

To this scope, we start with the global balance laws as it is used to do in any other problem in continuum mechanics. More specifically, we consider an elastic body with a propagating crack in its interior and postulate the balances for all relevant fields, included the configurational ones, for any arbitrary part of the body (Agiasofitou and Kalpakides 2003). In the presence of the crack, this procedure becomes much more complicated because of two reasons. Firstly, the involved fields are not continuous across the crack, even more, they may have a singularity at the crack tip and second, the underlying kinematics is more complicated due to the presence of the separate crack kinematics. In particular, the singularities at the crack tip make necessary to reformulate the transport and divergence theorems, which are indispensable for any localization process.

In literature, such a view can be found in the work of (Steinmann 2000) who presented balance laws in both the physical and material space for elastostatics of a smooth elastic body. Also, reports to equations, which can be considered as balance laws for a fractured body, appeared in (Maugin 1993, 1995; Dascalu and Maugin 1995; Gurtin and Podio-Guidugli 1996; Gurtin 2000; Kienzler and Herrmann 2000) but, to the best of our knowledge, up to now there is not a complete and consistent analysis in the spirit of modern continuum mechanics.

The global view adopted in this paper can shed light on the relationship between the configurational fields at the crack tip and the energy release rates as well as the connection between the first ones and the J and L integrals. For instance, starting from the pseudomomentum and energy equations, one can establish a connection between the energy release rate and the configurational force at the crack tip. This quantity is referred to by Maugin as global material force and it is directly related to the J–integral (Maugin 1993; Dascalu and Maugin 1995). One of our goal in this paper is to explore an analogous relation starting from the material angular momentum and energy equations. In this case, it is expected a connection between the configurational moment at the crack tip and the rotational energy release rate. Such a relation has been provided by (Maugin and Trimarco 1995) for the case where the defect is a disclination line.

Furthermore, (Golebiewska Herrmann and Herrmann 1981) considered the case of a stationary crack which rotates and they computed the rotational energy release rate. Also, (Eischen and Herrmann 1987) tried to connect the conservation (and balance) laws with the energy release rates and the J, L and M integrals. In these works, a straight stationary crack is considered and the rotational energy release rate emerges by a virtual rotation of the crack around its center. Although this is a very successful and meaningful manipulation (in the sense that the associated conservation law is coming from the invariance of the action functional under the group of rotations), it can not be related to a real situation of a propagating crack.

Looking for a more physical interpretation, the propagation of a crack along a curve of arbitrary curvature is considered in such a way that the linear and the angular velocity of the crack tip to be inserted. The balance laws are postulated and from the localization process the configurational fields at the crack tip naturally arise. Finally, these quantities are correlated with the energy release rates and the J and L integrals of fracture mechanics.

Although the crack propagation in a deformable body is a dissipative phenomenon, in this paper no mention is made to the second law of thermodynamics and to the subsequent discussion about constitutive relations.

In Section 2, some preliminaries concerning the proper kinematics for a cracked elastic body are presented. In Section 3, an abstract balance law is postulated, the conditions under which it is meaningful are examined and its consequences are extracted rigorously. The application of this procedure to the physical and configurational fields, related to the problem under study, is made in Sections 4 and 5, respectively. Finally, in Section 6, the obtained results are used to derive the relations between the energy release rates and the configurational fields at the crack tip.

2 PRELIMINARIES

Let $\mathcal{B}_R$ be the reference configuration containing a crack which is described by a smooth, non-intersecting curve C_R with the one end point to lie on the boundary of the body and the other one to be the crack tip, $\mathbf{Z}_0$. We consider that the crack evolves, not necessarily in straight direction, following the "motion" of the crack tip within the body. Thus at the time t, the crack is represented by a smooth curve $C(t)$ belonging to a material configuration $\mathcal{B}_t,\ t \in I \subset I\!R$, where I denotes a time interval. The only difference between the reference configuration $\mathcal{B}_R$ and the material configurations $\mathcal{B}_t$ lies in the different curve they contain. Certainly, it is required for $t_1 > t_2$ to imply $C(t_2) \subset C(t_1)$.

We focus now on the end point of the crack at time t, $\mathbf{Z}(t)$. We consider that $\mathbf{Z}(t)$ is a smooth, time dependent mapping, hence its derivative

$$\mathbf{V}(t) = \frac{d\mathbf{Z}}{dt} \tag{1}$$

provides the propagation velocity of the crack. Also, if we denote with $\mathbf{t}$ the tangent vector to the crack curve, we can write $\mathbf{V} = V\mathbf{t}$.
Taking the standard view of fracture mechanics, we consider a disc of radius ϵ centered at the crack tip $\mathbf{Z}(t)$ for any time t, denoted by $D_\epsilon(t)$:

$$D_\epsilon(t) = \{\mathbf{X} \in \mathcal{B}_t : |\mathbf{X} - \mathbf{Z}(t)| \leq \epsilon\}. \tag{2}$$

At the time t_0, the tip disc is given by:

$$D_{\epsilon 0} = \{\mathbf{Y} \in \mathcal{B}_R : |\mathbf{Y} - \mathbf{Z}_0| \leq \epsilon\}.$$

Notice here that $D_{\epsilon 0} \subset \mathcal{B}_R$ and $D_\epsilon(t) \subset \mathcal{B}_t$. Also, we will denote the part of the crack curve which lies on $D_\epsilon(t)$ with γ_D, i.e., $\gamma_D = D_\epsilon(t) \cap C(t)$.
Taking into account the crack tip evolution, we can establish a fictitious motion of the tip disc (Fig.1) in the following form

$$\mathbf{X} = X(\mathbf{Y}, t), \quad \mathbf{X} \in D_\epsilon(t), \quad \mathbf{Y} \in D_{\epsilon 0},\ t \in I. \tag{3}$$

Without any loss of generality, we assume that this "motion" is a rigid one (Gurtin 1981) and particularly, it is a simple translation which follows the crack tip evolution, that is

$$X(\mathbf{Y}, t) = \mathbf{Y} + \mathbf{Z}(t) - \mathbf{Z}_0, \quad \text{for all } \mathbf{Y} \in D_{\epsilon 0}. \tag{4}$$

It is obvious that every point of $D_{\epsilon 0}$ "moves" with the velocity of the crack tip, i.e.,

$$\mathbf{V}(\mathbf{Y}, t) = \frac{\partial X}{\partial t}(\mathbf{Y}, t) = \frac{d\mathbf{Z}}{dt} = \mathbf{V}(t), \quad \text{for all } \mathbf{Y} \in D_{\epsilon 0}. \tag{5}$$

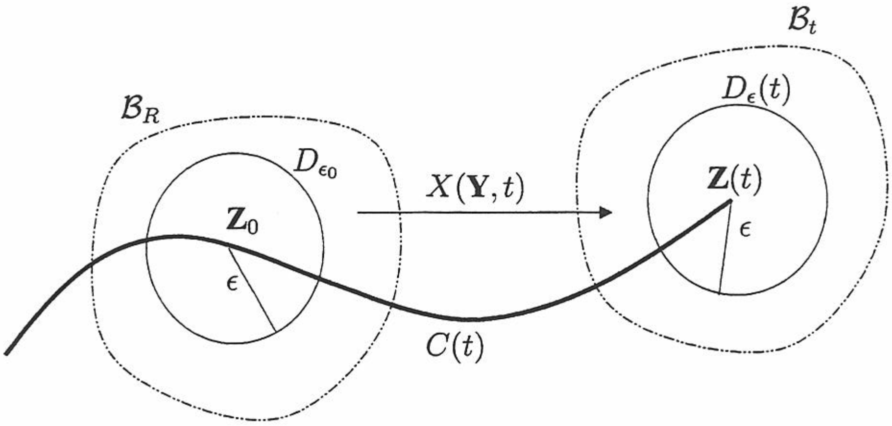

Figure 1: The motion of the tip disc

Consider now the physical motion

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \mathbf{x} \in B_t, \quad \mathbf{X} \in \mathcal{B}_t,\ t \in I \subset I\!R, \tag{6}$$

which is twice-differentiable for all $(\mathbf{X}, t) \in (\mathcal{B}_t \setminus C(t)) \times I$. Also, it is continuous along the crack curve $C(t) \setminus \mathbf{Z}(t)$, as we assume that the crack faces are in perfect contact. We observe that the material points $\mathbf{X} \in D_\epsilon(t)$ depend on t via the mapping X, while the material points $\mathbf{X} \in \mathcal{B}_t \setminus D_\epsilon(t)$ do not depend on t. Consequently, we can compose the mappings X and χ for all $\mathbf{X} \in D_\epsilon(t)$ to interpret both the crack evolution and the motion of the body in the physical space (Fig. 2). Note that this composition holds only for those $\mathbf{X}$ that belong to $D_\epsilon(t)$ at the time t. As a result, we can write for all $\mathbf{X} \in D_\epsilon(t)$

$$\widetilde{\chi} = \chi \circ X, \quad \mathbf{x} = \widetilde{\chi}(\mathbf{Y}, t) = \chi(X(\mathbf{Y}, t), t), \mathbf{Y} \in D_{\epsilon 0}. \tag{7}$$

The partial derivative of $\widetilde{\chi}$ with respect to time will be denoted by $\overset{\circ}{\mathbf{x}}$ and the following chain differentiation will hold

$$\overset{\circ}{\mathbf{x}} = \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}, t)\, \frac{\partial X}{\partial t}(\mathbf{Y}, t) + \frac{\partial \chi}{\partial t}(\mathbf{X}, t), \quad \text{for all} \quad \mathbf{X} \in D_\epsilon(t) \setminus \gamma_D. \tag{8}$$

While for all $\mathbf{X} \in \mathcal{B}_t \setminus D_\epsilon(t)$ away from the crack, it holds

$$\overset{\circ}{\mathbf{x}} = \frac{\partial \chi}{\partial t}(\mathbf{X}, t), \quad \text{for all} \quad \mathbf{X} \in \mathcal{B}_t \setminus D_\epsilon(t).$$

Denoting, as usually, with $\mathbf{F}(\mathbf{X}, t) = \partial \chi(\mathbf{X}, t)/\partial \mathbf{X}$ the deformation gradient and taking into account eq. (5), the equation (8) takes the form

$$\overset{\circ}{\mathbf{x}} = \mathbf{F}(\mathbf{X}, t)\mathbf{V}(t) + \dot{\mathbf{x}}(\mathbf{X}, t) =: \widetilde{\mathbf{V}}(\mathbf{X}, t), \quad \text{for all} \quad \mathbf{X} \in D_\epsilon(t) \setminus \gamma_D, \tag{9}$$

where $\dot{\mathbf{x}}(\mathbf{X}, t) = \partial \chi(\mathbf{X}, t)/\partial t$. Note that, from the above assumptions about the smoothness of x, we have that $\mathbf{F}(\mathbf{X}, t)$ and $\dot{\mathbf{x}}(\mathbf{X}, t)$ are continuous for $\mathbf{X} \in D_\epsilon(t) \setminus \gamma_D$. However, both $\mathbf{F}$ and $\dot{\mathbf{x}}$ are singular at the crack tip $\mathbf{Z}(t)$.

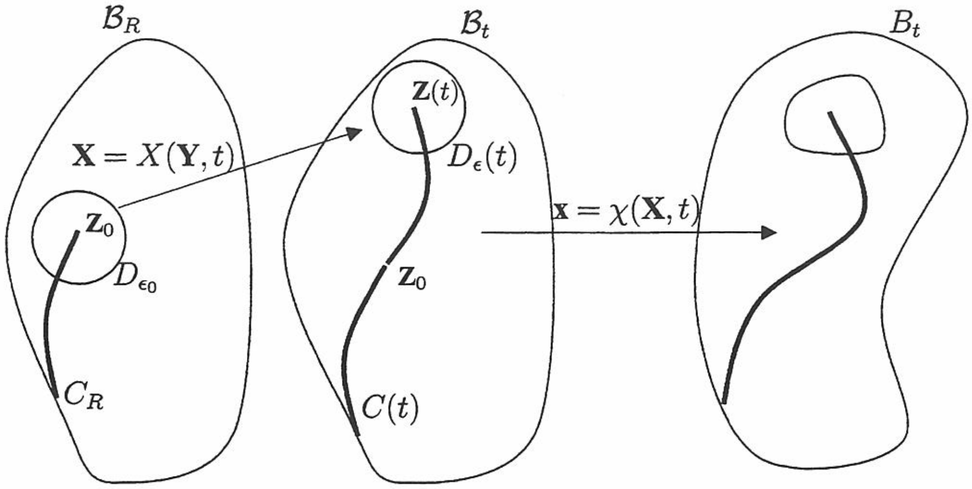

Figure 2: The total motion

The quantity $\widetilde{\mathbf{V}}(\mathbf{X},t)$ represents the velocity of the deformed tip disc accounting for the crack evolution velocity as well. Though $\widetilde{\mathbf{V}}(\mathbf{X},t)$ is defined with the aid of the fields $\mathbf{F}$ and $\dot{\mathbf{x}}$ which are singular at the crack tip, we would like $\widetilde{\mathbf{V}}$ to be smooth at the crack tip. Thus, taking the view of (Gurtin 2000, Gurtin and Podio-Guidugli 1996), we assume the existence of a bounded, time–dependent function $\widetilde{\mathbf{U}}(t)$ such that

$$\lim_{\mathbf{X}\to\mathbf{Z}(t)} \widetilde{\mathbf{V}}(\mathbf{X},t) = \widetilde{\mathbf{U}}(t), \text{ uniformly in } I. \tag{10}$$

Notice that the quantity $\tilde{\mathbf{U}}(t)$ represents the velocity of the deformed crack tip.

3 AN ABSTRACT BALANCE LAW FOR A CRACKED BODY

Let Ω be any smooth domain of the body in the material configuration $\mathcal{B}_t$. If the crack tip $\mathbf{Z}(t)$ is an interior point of Ω, then there exists a radius ϵ such that $D_\epsilon(t) \subset \Omega$. In this case, we will denote with Ω_ϵ the subset of Ω which is defined as follows (Fig. 3),

$$\Omega_\epsilon(t) = \Omega \setminus D_\epsilon(t) \ \text{ or } \ \Omega = \Omega_\epsilon(t) \cup D_\epsilon(t). \tag{11}$$

Notice that $\partial\Omega_\epsilon = \partial\Omega \cup \partial D_\epsilon(t)$. Also, the parts of the crack $C(t)$ contained in Ω_ϵ and Ω will be denoted by γ_ϵ and γ_Ω, respectively , that is

$$\gamma_\epsilon = C(t) \cap \Omega_\epsilon(t), \quad \gamma_\Omega = C(t) \cap \Omega.$$

In standard continuum mechanics, one has the freedom to formulate a global balance law either in the reference configuration or in the current configuration. In the proposed framework, there are three distinct configurations (Fig.2). We work on a material configuration $\mathcal{B}_t$, in which all the relevant fields should be defined. Let $\phi(\mathbf{X},t)$ be a scalar valued function defined in $\mathcal{B}_t$, representing some physical quantity, sufficiently smooth away from the crack tip and up to the crack $C(t)$ from either side, thus we let ϕ to have a singularity at the crack tip and to be discontinuous with finite jump along $C(t) \setminus \{\mathbf{Z}(t)\}$.

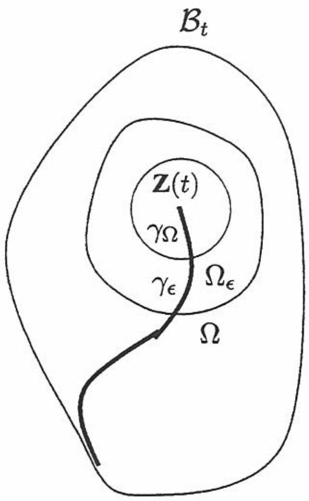

Figure 3: A domain Ω containing the crack tip

Taking the view of (Gurtin 2000), we will assume *the integrability of ϕ in the sense of Cauchy principal value*, i.e. for all $\Omega \in \mathcal{B}_t$

$$\int_{\Omega} \phi(\mathbf{X},t)\, dA = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \phi(\mathbf{X},t)\, dA. \tag{12}$$

Analogously, it holds for the line integral of a vector valued function $\mathbf{g}(\mathbf{X},t)$ along the curve γ_Ω in the sense

$$\int_{\gamma_\Omega} \mathbf{g}(\mathbf{X},t) \cdot \mathbf{n}\, dl = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \mathbf{g}(\mathbf{X},t) \cdot \mathbf{n}\, dl, \tag{13}$$

where $\mathbf{n}$ is the unit normal to $C(t)$. Hereafter, when we refer to the integrability of any function over Ω and γ_Ω, it will be meant in the sense of eqs. (12) and (13).
Next, we consider a *global balance law for the quantity ϕ* of the form

$$\frac{d}{dt} \int_{\Omega} \phi(\mathbf{X},t)\, dA = \int_{\partial\Omega} \mathbf{f}(\mathbf{X},t) \cdot \mathbf{N}\, dS + \int_{\Omega} h(\mathbf{X},t)\, dA + g(t), \tag{14}$$

where $\mathbf{N}$ is the outward unit normal to the boundary $\partial\Omega$ and $\mathbf{f}$ and h, are the flux and the source of ϕ, respectively. The time dependent function g represents the source of ϕ due to the crack evolution.
It is apparent that the integrability of ϕ is not enough to make eq. (14) meaningful. So, we must pose extra smoothness on the integrands. Denoting with $[\mathbf{f}]$ the jump of $\mathbf{f}$ across the crack, we assume the following conditions

C1 $h, \ \phi$ are integrable over Ω.

C2 $\displaystyle \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \frac{\partial \phi}{\partial t}(\mathbf{X},t)\, dA = \int_{\Omega} \frac{\partial \phi}{\partial t}(\mathbf{X},t)\, dA,$ uniformly in I.

C3 $\displaystyle \int_{\partial D_\epsilon} \phi(\mathbf{X},t)(\mathbf{V} \cdot \mathbf{N})\, dS$ converges uniformly in I as $\epsilon \to 0$.

C4 $\mathrm{Div}\, \mathbf{f}, \ [\mathbf{f}] \cdot \mathbf{n}$ are integrable over Ω and γ_Ω, respectively.

C5 $\displaystyle \int_{\partial D_\epsilon} \mathbf{f}(\mathbf{X},t) \cdot \mathbf{N}\, dS$ converges to a time dependent function as $\epsilon \to 0$.

One can prove the following statement (Agiasofitou and Kalpakides 2003)
Assume that the Conditions 1, 2 and 3 hold. Then, $\int_\Omega \phi(\mathbf{X},t)\, dA$ is a differentiable function of t. In addition, its derivative will be given by the relation

$$\frac{d}{dt} \int_{\Omega} \phi(\mathbf{X},t)\, dA = \lim_{\epsilon \to 0} \left(\frac{d}{dt} \int_{\Omega_\epsilon} \phi(\mathbf{X},t)\, dA \right). \tag{15}$$

The transport theorem and the divergence theorem for any domain Ω_ϵ, can be written, respectively

$$\frac{d}{dt} \int_{\Omega_\epsilon} \phi(\mathbf{X},t)\, dA = \int_{\Omega_\epsilon} \frac{\partial \phi(\mathbf{X},t)}{\partial t}\, dA - \int_{\partial D_\epsilon} \phi(\mathbf{X},t)(\mathbf{V} \cdot \mathbf{N})\, dS \tag{16}$$

and

$$\int_{\Omega_\epsilon} \text{Div}\, \mathbf{f}(\mathbf{X},t)\, dA = \int_{\partial\Omega} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\, dS - \int_{\partial D_\epsilon} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\, dS + \int_{\gamma_\epsilon} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl. \tag{17}$$

Using the Conditions 1-5 and the eqs. (16)-(17), one can prove the following versions for the transport theorem and divergence theorem appropriate for the problem under study (Agiasofitou and Kalpakides 2003)

$$\frac{d}{dt}\int_\Omega \phi(\mathbf{X},t)\, dA = \int_\Omega \frac{\partial\phi}{\partial t}(\mathbf{X},t)\, dA - \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \phi(\mathbf{X},t)(\mathbf{V}\cdot\mathbf{N})\, dS \tag{18}$$

and

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{N}\, dS = \int_\Omega \text{Div}\, \mathbf{f}\, dA + \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\, dS - \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl. \tag{19}$$

Inserting eqs. (18) and (19) into eq. (14), we obtain

$$\int_\Omega \left(\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dA + \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl - \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \left(\phi(\mathbf{X},t)(\mathbf{V}\cdot\mathbf{N}) + \mathbf{f}(\mathbf{X},t)\cdot\mathbf{N}\right) dS - g(t) = 0, \tag{20}$$

for all Ω containing the crack tip.
We remark that in the case where Ω does not contain the crack tip and any part of the crack, eq. (20) takes the simpler form

$$\int_\Omega \left(\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dA = 0. \tag{21}$$

Thus, due to the arbitrariness of Ω, we conclude that

$$\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t) = 0, \quad \text{for all } t\in I,\ \mathbf{X}\in\mathcal{B}_t\setminus C(t). \tag{22}$$

Similarly, we can consider Ω containing a part of the crack apart from the crack tip. In this case, the global balance law (eq. (20)) takes the form

$$\int_\Omega \left(\frac{\partial\phi(\mathbf{X},t)}{\partial t} - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t)\right) dA + \int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl = 0. \tag{23}$$

However, $\int_\Omega (\partial\phi(\mathbf{X},t)/\partial t - \text{Div}\, \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t))\ dA = 0$, because its integrand is zero almost everywhere due to eq. (22), i.e., it is zero everywhere apart from the crack line γ_Ω, which is a set of measure zero in Ω. Consequently, eq. (23) gives

$$\int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)]\cdot\mathbf{n}\, dl = 0, \tag{24}$$

for all γ_Ω which do not contain the crack tip. Thus, we obtain

$$[\mathbf{f}(\mathbf{X},t)] \cdot \mathbf{n} = 0, \quad \text{for all } \; t \in I, \; \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}. \tag{25}$$

In the same line of argument, we consider arbitrary Ω which contains the whole crack. In this case, we must use the complete form of eq. (20). Taking into account the results provided by eqs. (22) and (25), we remark that the integrands of the first two terms of eq. (20) vanish almost everywhere in any Ω_ϵ and any γ_ϵ, respectively, thus we can write

$$\int_{\Omega_\epsilon} \left(\frac{\partial \phi(\mathbf{X},t)}{\partial t} - \operatorname{Div} \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t) \right) dA = 0,$$
$$\int_{\gamma_\epsilon} [\mathbf{f}(\mathbf{X},t)] \cdot \mathbf{n} \, dl = 0,$$

for all $\epsilon > 0$.
Thus, recalling the sense of integrability given by eqs. (12) and (13), we conclude that

$$\int_{\Omega} \left(\frac{\partial \phi(\mathbf{X},t)}{\partial t} - \operatorname{Div} \mathbf{f}(\mathbf{X},t) - h(\mathbf{X},t) \right) dA = 0, \tag{26}$$

$$\int_{\gamma_\Omega} [\mathbf{f}(\mathbf{X},t)] \cdot \mathbf{n} \, dl = 0, \tag{27}$$

for all Ω and γ_Ω, even they contain the crack tip. Finally, we obtain the localization of the balance law at the crack tip as follows:

$$g(t) = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\phi(\mathbf{X},t)(\mathbf{V} \cdot \mathbf{N}) + \mathbf{f}(\mathbf{X},t) \cdot \mathbf{N}) dS, \quad \text{for all } \; t \in I. \tag{28}$$

To sum up, the requirement that the balance law (14) holds for all $\Omega \in \mathcal{B}_t$ implies the local equations (22), (25) and (28), that is,

$$\begin{aligned} &\frac{\partial \phi}{\partial t} - \operatorname{Div} \mathbf{f} - h = 0, \qquad \text{for all } \; t \in I, \; \mathbf{X} \in \mathcal{B}_t \setminus C(t), \\ &[\mathbf{f}] \cdot \mathbf{n} = 0, \qquad \text{for all } \; t \in I, \; \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \\ &g(t) = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\phi(\mathbf{V} \cdot \mathbf{N}) + \mathbf{f} \cdot \mathbf{N}) dS, \qquad \text{for all } \; t \in I. \end{aligned} \tag{29}$$

4 BALANCE LAWS IN THE PHYSICAL SPACE

Throughout this and the next section, we assume that each field inserted in a global balance law at the position of the abstract functions Φ, $\mathbf{f}$ and h will enjoy the corresponding smoothness specified in the previous section.

4.1 The balances of mass, momentum and angular momentum

We assume that there are no sources of mass, momentum and angular momentum, due to the crack evolution. Thus, we accept that, apart from the energy, the crack evolution does not intervene directly in the balance of the physical fields. Nevertheless, we expect a new relation at the crack tip due to the singularities of the physical fields. We denote with ρ and $\mathbf{T}$ the mass density in the material configuration and the Piola-Kirchhoff stress tensor, respectively. Also, the position vector of $\mathbf{x}$ is denoted with $\mathbf{r} = \mathbf{x} - \mathbf{0}$. As in the standard continuum mechanics, it is postulated that the mass, the momentum and the angular momentum fulfil the following relations

$$\frac{d}{dt}\int_{\Omega} \rho(\mathbf{X}, t)\, dA = 0, \tag{30}$$

$$\frac{d}{dt}\int_{\Omega} \rho\dot{\mathbf{x}}\, dA = \int_{\partial\Omega} \mathbf{TN}\, dS, \tag{31}$$

$$\frac{d}{dt}\int_{\Omega} \mathbf{r} \times \rho\dot{\mathbf{x}}\, dA = \int_{\partial\Omega} (\mathbf{r} \times \mathbf{T})\mathbf{N}\, dS, \tag{32}$$

for every part Ω of $\mathcal{B}_t$ and for every t in some interval I.
The local form of the balances (30-32) outside the crack are extracted from eq. $(29)_1$

$$\frac{\partial\rho(\mathbf{X}, t)}{\partial t} = 0 \Rightarrow \rho = \rho(\mathbf{X}), \tag{33}$$

$$\frac{\partial}{\partial t}(\rho\dot{\mathbf{x}}) - \mathrm{Div}\mathbf{T} = 0, \tag{34}$$

$$\frac{\partial}{\partial t}(\mathbf{r} \times \rho\dot{\mathbf{x}}) - \mathrm{Div}(\mathbf{r} \times \mathbf{T}) = 0, \tag{35}$$

for all $t \in I$, $\mathbf{X} \in \mathcal{B}_t \setminus C(t)$.
Moreover, the localization process gives the following jump conditions (see eq. $(29)_2$) along the crack curve

$$[\mathbf{T}]\mathbf{n} = 0, \tag{36}$$

$$[\mathbf{r} \times \mathbf{T}]\mathbf{n} = 0, \tag{37}$$

for all $t \in I$, $\mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}$. The above local equations and jump conditions do not differ from the corresponding ones holding for any smooth elastic body with a material surface of discontinuity within it. Recalling that the motion $\chi(\mathbf{X}, t)$ is continuous along the crack $C(t)$ (hence, $\mathbf{r}$ is continuous as well), we easily conclude that the condition (37) follows from the jump condition (36).
The new results of the proposed approach concern the relations holding at the crack tip are derived from$(29)_3$ as follows

$$\lim_{\epsilon \to 0}\int_{\partial D_\epsilon} \rho(\mathbf{V} \cdot \mathbf{N})\, dS = 0, \tag{38}$$

$$\lim_{\epsilon \to 0}\int_{\partial D_\epsilon} (\rho\dot{\mathbf{x}}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{TN})\, dS = 0, \tag{39}$$

$$\lim_{\epsilon \to 0}\int_{\partial D_\epsilon} \mathbf{r} \times (\rho\dot{\mathbf{x}}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{TN})\, dS = 0, \tag{40}$$

for all $t \in I$.
Equation (38) shows that the rate of mass flow through ∂D_ϵ vanishes, when the boundary shrinks onto the crack tip. Eqs. (39) and (40) represent the balance of linear momentum

and angular momentum at the crack tip, respectively. Adopting the *standard momentum condition* of (Gurtin 2000), that is,

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \rho\dot{\mathbf{x}}\otimes\mathbf{N}\,dS = 0,$$

we take from eq. (39)

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})\,dS = 0, \quad \lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \mathbf{TN}\,dS = 0.$$

4.2 The balance of Energy

Unlike the balances for mass, linear momentum and angular momentum, we assume that the balance of energy is directly influenced by the crack growth. This is quite reasonable because the crack propagation is a dissipative phenomenon, that is to say, the growth of the crack consumes a part of the energy given by the applied forces. Hence, an energy source term describing the total dissipation rate of the body, denoted here by $\Phi(t)$, must be added in the energy balance. Thus, the global balance law for energy can be postulated as

$$\frac{d}{dt}\int_\Omega (W+K)\,dA = \int_{\partial\Omega} \mathbf{TN}\cdot\dot{\mathbf{x}}\,dS - \Phi, \quad \text{for all } t\in I,\ \Omega\in\mathcal{B}_t, \tag{41}$$

where W is the elastic energy density and K is the kinetic energy density, both are defined per unit volume in material configuration.

Localizing eq. (41), we obtain (see eq. (29))

$$\frac{\partial}{\partial t}(W+K) - \text{Div}(\mathbf{T}^T\dot{\mathbf{x}}) = 0,\ \forall\, t\in I,\ \mathbf{X}\in\mathcal{B}_t\setminus C(t), \tag{42}$$

$$[\mathbf{T}^T\dot{\mathbf{x}}]\cdot\mathbf{n} = 0,\ \forall\, t\in I,\ \mathbf{X}\in C(t)\setminus\{\mathbf{Z}(t)\}, \tag{43}$$

$$\Phi = \lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\big((W+K)(\mathbf{V}\cdot\mathbf{N}) + \mathbf{T}^T\dot{\mathbf{x}}\cdot\mathbf{N}\big)dS,\ \forall\, t\in I. \tag{44}$$

It is obvious that eqs. (42) and (43) are the local energy equation and the associated jump condition along the crack, respectively. Also, eq. (44) is the energy flow out of the body and into the crack tip per unit time and if it be divided by the crack propagation velocity V, it will give the well-known, in fracture literature (Freund 1981), dynamic energy release rate G, i.e.,

$$G = \Phi/V. \tag{45}$$

5 BALANCE LAWS IN THE MATERIAL SPACE

The balances which we are dealt with in the last section do not exhaust all the relevant quantities involved in our problem. We must further consider balances for the configurational fields, that is, the pseudomomentum and the material angular momentum.

5.1 The balance of pseudomomentum

We introduce now the pseudomomentum (or material momentum)

$$\mathcal{P}(\mathbf{X},t) = -\rho\mathbf{F}^T\dot{\mathbf{x}},\ t\in I,\ \mathbf{X}\in\mathcal{B}_t\setminus C(t), \tag{46}$$

a quantity analogous to the physical momentum, concerning changes within the material structure. In a Hamiltonian framework, the pseudomomentum is the dual quantity to the velocity of the inverse motion function, like the physical momentum is the dual of the standard velocity of the body (Maugin and Kalpakides 2002). The contributors to the balance of pseudomomentum will be the material or configurational forces (Maugin 1993). Considering both at a distance and at contact configurational forces, we introduce the configurational body forces $\tilde{\mathbf{f}}$ (source term) and the configurational stress tensor $\mathbf{b}$ (flux term), respectively . Moreover, we consider a pseudomomentum source term, that is a material force, $\mathcal{F} = \mathcal{F}(t)$, produced by the crack evolution. After all these considerations, we postulate the balance law for the pseudomomentum

$$\frac{d}{dt}\int_{\Omega} \mathcal{P}\, dA = \int_{\partial\Omega} \mathbf{bN}\, dS + \int_{\Omega} \tilde{\mathbf{f}}\, dA + \mathcal{F},\ \forall t \in I,\ \forall\Omega \in \mathcal{B}_t. \tag{47}$$

The local equations, obtained by eq. (47), are given (see eqs. (29)) as follows

$$\frac{\partial \mathcal{P}}{\partial t} - \mathrm{Div}\mathbf{b} - \tilde{\mathbf{f}} = 0,\ \ \forall\, t \in I,\ \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{48}$$

$$[\mathbf{b}]\mathbf{n} = 0,\ \ \forall\, t \in I,\ \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\}, \tag{49}$$

$$\mathcal{F} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} (\mathcal{P}(\mathbf{V}\cdot\mathbf{N}) + \mathbf{bN})\, dS,\ \forall\, t \in I. \tag{50}$$

Eq. (48) is the equation of pseudomomentum, which holds in the smooth part of the body and eq. (49) is the associated jump condition. In addition, eq. (50) represents the material force at the crack tip, which drives the crack evolution. Thus, the quantity $\mathcal{F}$ should be directly related to the energy release rate, G. Also, in the static case it holds the following relation

$$\mathcal{F} = -\lim_{\epsilon \to 0} \mathbf{J}(\epsilon),$$

where $\mathbf{J}$ is the well–known $\mathbf{J}$ -integral of Rice (Budiansky and Rice 1973).
In the absence of a crack or any other rearrangement in the material configuration, the last term in the pseudomomentum balance law vanishes and eq. (48) holds all over the body as a simple identity for the solution of the standard elastic problem. In other words, eqs. (48–50) do not make sense in the standard continuum mechanics, where only the motion in physical space is considered. Thus, the balance law (47) must be considered when one studies any kind of evolution of structural defects. From this point of view, it is a configurational balance law.

Remark 1: One can enrich the balance law (47) by considering an additional term of the form $\int_{\gamma_\Omega} \mathbf{g}^l\, dl$, accounting for configurational forces acting along the crack curve (Gurtin 2000). In that case, the localization process provides

$$[\mathbf{b}]\mathbf{n} + \mathbf{g}^l = 0,$$

instead of eq. (49).

Remark 2: The flux term $\mathbf{b}$, like the Piola–Kirchhoff stress tensor in standard continuum mechanics, needs a constitutive relation to be further determined. Because the constitutive relations are out of the scope of the present procedure, we adopt without reasoning the relation

$$\mathbf{b} = (W - \frac{1}{2}\rho\dot{\mathbf{x}}^2)\mathbf{I} - \mathbf{F}^T\mathbf{T}, \tag{51}$$

that is, the Eshelby stress tensor for the dynamical case (Eshelby 1995). For the derivation and a discussion about this relationship, viewed as a constitutive relation, we refer to the works of (Gurtin 2000) and (Podio-Guidugli 2002). As concerns the term $\tilde{\mathbf{f}}$, we consider it as a distributed body material force, produced by the material inhomogeneities (Maugin 1993). Moreover, the pseudomomentum source term $\mathcal{F}$, produced by the crack evolution, is referred to by (Maugin 1993; Dascalu and Maugin 1995) as *global material force* and by (Gurtin 2000) as *tip traction*.

5.2 The balance of material angular momentum

We proceed to the balance of the material angular momentum, that is, the moment of pseudomomentum, $\mathbf{R} \times \mathcal{P}$, where $\mathbf{R} = \mathbf{X} - \mathbf{0}$ is the position vector of $\mathbf{X}$. The rest contributors to this law should be the moment of material contact and material body forces. Moreover, we consider a material angular momentum source term, $\mathcal{M} = \mathcal{M}(t)$ due to the presence of the crack. We postulate:

$$\frac{d}{dt}\int_{\Omega} \mathbf{R} \times \mathcal{P}\, dA = \int_{\partial\Omega} \mathbf{R} \times \mathbf{bN}\, dS + \int_{\Omega} \mathbf{R} \times \tilde{\mathbf{f}}\, dA + \int_{\Omega} \mathbf{g}\, dA + \mathcal{M}, \qquad \forall t \in I,\ \forall \Omega \in \mathcal{B}_t, \tag{52}$$

where $\mathbf{g}(\mathbf{X}, t)$ is a vector field describing the distribution of material couples within the body.

The localization of eq. (52) provides

$$\frac{\partial(\mathbf{R} \times \mathcal{P})}{\partial t} - \mathrm{Div}(\mathbf{R} \times \mathbf{b}) - \mathbf{R} \times \tilde{\mathbf{f}} - \mathbf{g} = 0, \ \ \forall t \in I,\ \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{53}$$

$$\mathcal{M} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\mathbf{R} \times (\mathcal{P}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{bN})\big)\, dS, \ \ \forall t \in I \tag{54}$$

and the associated jump condition

$$[\mathbf{R} \times \mathbf{b}]\mathbf{n} = 0, \quad \forall\, t \in I,\ \forall\, \mathbf{X} \in C(t) \setminus \{\mathbf{Z}(t)\},$$

which holds identically due to the continuity of $\mathbf{X}$ and the jump condition (49).

Eq. (53) is the equation of material angular momentum, which holds in the bulk of the body. If $\tilde{\mathbf{f}} = \mathbf{0}$ and $\mathbf{g} = \mathbf{0}$, then it coincides with the corresponding one of (Golebiewska Herrmann 1982).

If we take into account the equation of pseudomomentum (48), then eq. (53) gives the following relation

$$\mathbf{g} = \mathbf{axl b} \ \text{ or } \ g_A = -e_{ABC} b_{BC}, \tag{55}$$

where $\mathbf{axl b}$ denotes the axial vector of $\mathbf{b}$ (Chadwick 1976). Finally, eq. (53) is written as follows

$$\frac{\partial(\mathbf{R} \times \mathcal{P})}{\partial t} - \mathrm{Div}(\mathbf{R} \times \mathbf{b}) - \mathbf{R} \times \tilde{\mathbf{f}} - 2\mathbf{axl b} = 0, \ \ \forall t \in I,\ \mathbf{X} \in \mathcal{B}_t \setminus C(t), \tag{56}$$

which is in accordance with the corresponding one of (Steinmann 2000) for the static case.

Remark 3: If the material is homogeneous and isotropic, then the Eshelby stress tensor $\mathbf{b}$ is symmetric (Steinmann 2000; Kalpakides and Agiasofitou 2002), which means that $\mathrm{axl}\mathbf{b} = \mathbf{0}$, so eq. (55) gives $\mathbf{g} = \mathbf{0}$.

Remark 4: Adopting the existence of configurational forces distributed along the crack curve as we did in Remark 1, we obtain the jump condition

$$[\mathbf{R} \times \mathbf{b}]\mathbf{n} + \mathbf{R} \times \mathbf{g}^{l} = 0.$$

Furthermore, equation (54) gives the form of the *configurational moment $\mathcal{M}$ at the crack tip*. Particularly,

$$\mathcal{M}(t) = -\lim_{\epsilon \to 0} \mathcal{M}_{\epsilon}(t), \quad \forall t \in I,$$

where

$$\mathcal{M}_{\epsilon}(t) = \int_{\partial D_{\epsilon}} \big(\mathbf{R} \times (\mathcal{P}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{b}\mathbf{N})\big)\, dS. \tag{57}$$

The physical interpretation of $\mathcal{M}_{\epsilon}$ and its possible connection with the L–integral will be examined in the next subsection.

5.3 The configurational moment and the L–integral

It is worth noting that the Eshelby stress tensor used in fracture mechanics literature differs from the one used here. In fracture mechanics, the tensor $\mathbf{b}$ is defined with the aid of the displacement field $\mathbf{u}(\mathbf{X}, t)$, whereas in our analysis it is defined with the aid of the motion mapping $\mathbf{x}(\mathbf{X}, t)$ (see the relation (51)). If we introduce in eq. (51) the displacement field $\mathbf{u}$, we take

$$\mathbf{b} = (W - \frac{1}{2}\rho\dot{\mathbf{x}}^2)\mathbf{I} - \mathbf{F}^T\mathbf{T} = (W - \frac{1}{2}\rho\dot{\mathbf{u}}^2)\mathbf{I} - (\nabla\mathbf{u})^T\mathbf{T} - \tilde{\mathbf{I}}^T\mathbf{T},$$

where $\tilde{\mathbf{I}}$ denotes the two point unit tensor (or the shifter δ_{iA} in a coordinate system). Then, we can write

$$\mathbf{b} = \mathbf{b}^u - \tilde{\mathbf{I}}^T\mathbf{T}, \tag{58}$$

where

$$\mathbf{b}^u = (W - \frac{1}{2}\rho\dot{\mathbf{u}}^2)\mathbf{I} - (\nabla\mathbf{u})^T\mathbf{T}.$$

Notice that using $\mathbf{b}^u$ instead of $\mathbf{b}$ in pseudomomentum equation and neglecting the configurational body forces $\tilde{\mathbf{f}}$, we obtain, in virtue of eq. (34), an equation of the same form

$$\frac{\partial \mathcal{P}^u}{\partial t} - \mathrm{Div}\mathbf{b}^u = 0,$$

where

$$\mathcal{P}^u = -\rho(\nabla\mathbf{u})^T\dot{\mathbf{u}}. \tag{59}$$

However, under the same manipulation the material angular momentum equation does not retain its form. Indeed, inserting eqs. (58) and (59) into eq. (53), neglecting $\tilde{\mathbf{f}}$, $\mathbf{g}$ and taking into account the equation of angular momentum, i.e. eq. (35), we obtain

$$\frac{\partial}{\partial t}(\mathbf{R} \times \mathcal{P}^u + \mathbf{u} \times \rho\dot{\mathbf{u}}) - \mathrm{Div}\,(\mathbf{R} \times \mathbf{b}^u + \mathbf{u} \times \mathbf{T}) = 0. \tag{60}$$

Consequently, eq. (60) must be used in any comparison of the present results with the corresponding ones in the linear fracture mechanics. Indeed, if $\mathbf{R}$ and $\mathbf{T}$ are replaced by the spatial coordinates $\mathbf{x}$ and the Caushy stress tensor $\boldsymbol{\sigma}$, respectively and ρ and W are defined per unit deformed volume, then eq. (60) coincides with the corresponding one of (Fletcher 1975), for a linear, homogeneous and isotropic elastic body in the absence of body forces.
In addition, doing the same replacements in the integral given by eq. (57), the latter becomes

$$\mathcal{M}_\epsilon(t) = \int_{\partial D_\epsilon} \Big((\mathbf{R} \times \mathcal{P}^u + \mathbf{u} \times \rho \dot{\mathbf{u}})(\mathbf{V} \cdot \mathbf{N}) + (\mathbf{R} \times \mathbf{b}^u + \mathbf{u} \times \mathbf{T})\,\mathbf{N} \Big)\, dS \qquad (61)$$

In the static case, apart from the contour of integration, the integral $\mathcal{M}_\epsilon$ reduces to the L–integral, as it was given by (Knowles and Sternberg 1972) and (Steinmann 2000) for a nonlinear, homogeneous and isotropic elastic material. Therefore, an integral having the same integrand with $\mathcal{M}_\epsilon$ along an integration path encircling the total crack can be considered as a generalization of L–integral in the dynamical, non–linear case. It is important to remark that, in fracture mechanics literature, the path of the L–integral includes the whole crack, while in our analysis the path ∂D_ϵ is limited around the crack tip. This, on the one hand, justifies the term "configurational moment at the crack tip" and on the other, provides possibly an alternative physical interpretation of $\mathcal{M}$ and $\mathcal{M}_\epsilon$. More specifically, one can conjecture that the quantity $\mathcal{M}$ is related to the tendency of the crack tip (and as a result of the crack) to turn, while the usual interpretation (for instance, see (Golebiewska Herrmann and Herrmann 1981)) of the L–integral concerns the tendency of a stationary straight crack to rotate, as a whole, with respect to its center.

6 THE ENERGY RELEASE RATES AND THE CONFIGURATIONAL FIELDS

In this section, expressions for the energy release rates will be derived. Particularly, the relationship between the dynamical energy release rate with the configurational force at the crack tip as well as the relationship of the rotational energy release rate with the configurational moment at the crack tip, will be established.

6.1 The energy release rate and the configurational force

We start with the expression for the rate of energy dissipation, i.e., eq. (44):

$$\begin{aligned} \Phi &= \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big[(W + K)(\mathbf{V} \cdot \mathbf{N}) + \mathbf{T}^T \dot{\mathbf{x}} \cdot \mathbf{N} \big] dS \\ &= \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big[(W - K)(\mathbf{V} \cdot \mathbf{N}) \big] dS + \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \dot{\mathbf{x}} \big[\rho \dot{\mathbf{x}}(\mathbf{V} \cdot \mathbf{N}) + \mathbf{T}\mathbf{N} \big] dS. \end{aligned}$$

Recalling relation (9), the above equation can be written as

$$\begin{aligned}
\Phi &= \lim_{\epsilon\to 0}\int_{\partial D_\epsilon}[(W-K)(\mathbf{V}\cdot\mathbf{N})-\mathbf{FV}(\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN})]\,dS \\
&\qquad\qquad+\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{V}}\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS \\
&= \lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\mathbf{V}\cdot\left[\left((W-K)\mathbf{I}-\mathbf{F}^T\mathbf{T}\right)\mathbf{N}-\rho\mathbf{F}^T\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})\right]dS \\
&\qquad\qquad+\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{V}}\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS.
\end{aligned}\tag{62}$$

Due to eqs. (46) and (51), eq. (62) becomes

$$\begin{aligned}
\Phi = \lim_{\epsilon\to 0}&\int_{\partial D_\epsilon}\mathbf{V}\cdot[\mathbf{bN}+\boldsymbol{\mathcal{P}}(\mathbf{V}\cdot\mathbf{N})]\,dS \\
&+\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{V}}\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS.
\end{aligned}\tag{63}$$

One can prove that under specific assumptions the second term in eq. (63) vanishes. The essential step to this end is to prove the following

Proposition: *Assume that*

$$\int_{\partial D_\epsilon}|\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}|dS,\quad \text{is bounded as}\quad \epsilon\to 0.\tag{64}$$

Then the following convergence holds

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\left[\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)\right]\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS=0.\tag{65}$$

PROOF: We have

$$\begin{aligned}
&|\int_{\partial D_\epsilon}\left[\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)\right]\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\;dS| \\
&\le\int_{\partial D_\epsilon}\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)|\;|\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}|\;dS \\
&\le\sup_{\mathbf{X}\in\partial D_\epsilon}\left(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)|\right)\int_{\partial D_\epsilon}|\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}|dS.
\end{aligned}\tag{66}$$

On the other hand, the condition given by eq. (10) means that

$$\lim_{\mathbf{X}\to\mathbf{Z}(t)}(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)|)=0.$$

From this convergence, taking into account that the boundary ∂D_ϵ shrinks onto $\mathbf{Z}(t)$ as $\epsilon\to 0$, it is implied that the following convergence holds as well

$$\lim_{\epsilon\to 0}\left(\sup_{\mathbf{X}\in\partial D_\epsilon}(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)|)\right)=0.$$

The latter jointly with the assumption (64) gives

$$\lim_{\epsilon\to 0}\left[\sup_{\mathbf{X}\in\partial D_\epsilon}\left(\sup_{t\in I}|\widetilde{\mathbf{V}}(\mathbf{X},t)-\widetilde{\mathbf{U}}(t)|\right)\int_{\partial D_\epsilon}|\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}|dS\right]=0,$$

which, in virtue of inequality (66), completes the proof.
Next, from eq. (65) invoking the balance of physical momentum at the crack tip i.e., eq. (39), we conclude that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{V}}(\mathbf{X},t)\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS\\ &=\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\widetilde{\mathbf{U}}(t)\cdot[\rho\dot{\mathbf{x}}(\mathbf{V}\cdot\mathbf{N})+\mathbf{TN}]\,dS=0.\end{aligned}\tag{67}$$

Therefore, taking into account eq. (67), the energy flux at the crack tip, i.e., eq. (63) becomes

$$\Phi=\mathbf{V}\cdot\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}(\mathbf{bN}+\mathcal{P}(\mathbf{V}\cdot\mathbf{N}))dS$$

or, due to eq. (50),

$$\Phi=-\mathbf{V}\cdot\boldsymbol{\mathcal{F}}.\tag{68}$$

Using the definition (45) for the energy release rate, eq. (68) gives the following result

$$G=-\boldsymbol{\mathcal{F}}\cdot\mathbf{t},\tag{69}$$

which confirms that the energy release rate G is the crack driving force.

Remark 5: In the preceding analysis, two relations, which can be viewed as constraints on the singularity order for the velocity and the stress fields at the crack tip, have been arisen. These relations are the condition (64) and the equation (39). Suppose that the independent variables of a function $f(\mathbf{X},t)$ (say f be the velocity or the stress tensor) can be separated as

$$f(\mathbf{X},t)=g(r)h(\theta,t),$$

where $r=|\mathbf{X}-\mathbf{Z}(t)|$ and θ are the distance and the angle, respectively, in a polar coordinate system with its origin at the crack tip. Then, assuming that $g(r)=O(r^p),\ p\geq -1$ is sufficient to assure that the condition (64) holds. Furthermore, assuming that $g(r)=O(r^p),\ p>-1$, we obtain that eq. (39) holds, as well. However, it is well known that for a linear elastic, cracked body, both the near tip velocity and stress fields are of order $O(r^{-\frac{1}{2}})$ (Freund 1981).

6.2 The rotational energy release rate and the configurational moment at the crack tip

In this section, we will show that a relationship between the rotational energy release rate and the configurational fields at the crack tip can be established. We start with the relation (57) which can be written as

$$\boldsymbol{\mathcal{M}}=-\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\mathbf{R}\times\mathcal{C}\mathbf{N}\,dS,\tag{70}$$

where

$$\mathcal{C} = \mathcal{P} \otimes \mathbf{V} + \mathbf{b}^T. \tag{71}$$

Considering that the crack evolves along an arbitrary smooth curve, we introduce the angular velocity of the crack tip

$$\boldsymbol{\omega}(t) = \omega \mathbf{m}, \tag{72}$$

where $\mathbf{m} = \mathbf{t} \times \mathbf{n}$ is the unit normal vector to the plane of the crack. Furthermore, denoting with $\mathbf{a} = a\ \mathbf{n}$ the instantaneous radius of curvature ($\mathbf{n}$ the unit normal to the curve), we can write

$$\omega = \frac{V}{a}. \tag{73}$$

Also, we denote with $\mathbf{R}_Z = \mathbf{Z} - \mathbf{0} = R_Z \mathbf{e}$ the position vector of the crack tip. Then, we can write (see Fig.4)

$$\mathbf{R}_Z = \mathbf{R} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = -\epsilon \mathbf{N}.$$

Therefore, the configurational moment $\mathcal{M}$ becomes

$$\mathcal{M} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\mathbf{R}_Z \times \mathcal{C}\mathbf{N}\big)\, dS + \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\boldsymbol{\epsilon} \times \mathcal{C}\mathbf{N}\big)\, dS. \tag{74}$$

One can prove that the last term in eq. (74) vanishes under a particular condition. Indeed,

$$\int_{\partial D_\epsilon} \big(\boldsymbol{\epsilon} \times \mathcal{C}\mathbf{N}\big)\, dS = -\int_{\partial D_\epsilon} \big(\epsilon \mathbf{N} \times \mathcal{C}\mathbf{N}\big)\, dS = -\epsilon \int_{\partial D_\epsilon} \big(\mathbf{N} \times \mathcal{C}\mathbf{N}\big)\, dS.$$

In addition, it holds

$$\Big|\int_{\partial D_\epsilon} \big(\mathbf{N} \times \mathcal{C}\mathbf{N}\big)\, dS\Big| \leq \int_{\partial D_\epsilon} \big(|\mathbf{N}| \times |\mathcal{C}\mathbf{N}|\big)\, dS = \int_{\partial D_\epsilon} |\mathcal{C}\mathbf{N}|\, dS.$$

Thus, assuming that the integral $\int_{\partial D_\epsilon} |\mathcal{C}\mathbf{N}|\, dS$ is bounded as $\epsilon \to 0$, we obtain that the integral $\int_{\partial D_\epsilon} \big(\boldsymbol{\epsilon} \times \mathcal{C}\mathbf{N}\big)\, dS$ vanishes as $\epsilon \to 0$.
Consequently, the expression for $\mathcal{M}$ (eq. (74)) becomes

$$\mathcal{M} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \big(\mathbf{R}_Z \times \mathcal{C}\mathbf{N}\big)\, dS = -\mathbf{R}_Z \times \left(\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \mathcal{C}\mathbf{N}\, dS\right) \tag{75}$$

Notice that using the relation (71), the quantity $\mathcal{F}$ is written

$$\mathcal{F} = -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \mathcal{C}\mathbf{N}\, dS.$$

So, $\mathcal{M}$ is given by the following simple formula

$$\mathcal{M} = \mathbf{R}_Z \times \mathcal{F}, \tag{76}$$

which confirms the term configurational moment at the crack tip, since, essentially, it is the moment of the configurational force at the crack tip.
In addition, if θ is the angle from the $\mathbf{t}$-axis to the $\mathbf{e}$-axis, then $\mathcal{M}$ is written in terms of $\mathcal{F}$ as follows

$$\mathcal{M} = R_Z\big(\cos\theta\ (\mathcal{F} \cdot \mathbf{n}) - \sin\theta(\mathcal{F} \cdot \mathbf{t})\big)\mathbf{m}. \tag{77}$$

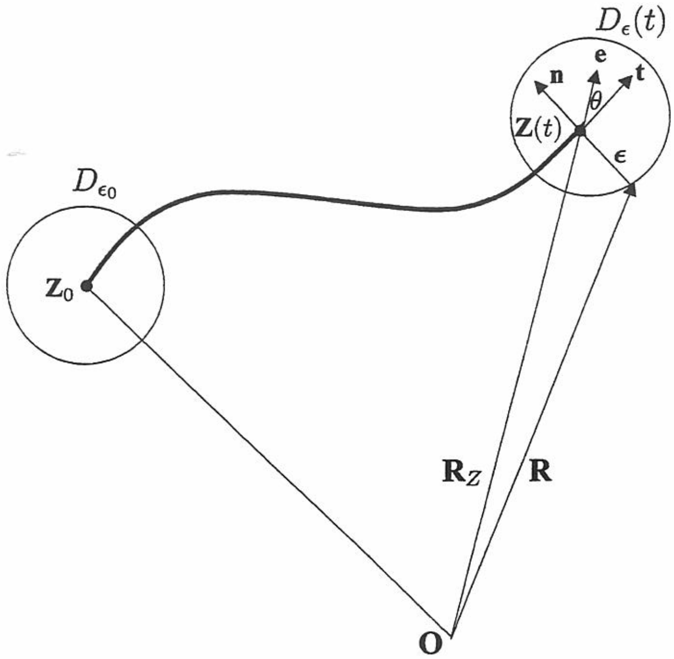

Figure 4: Some geometrical characteristics of the crack

Next, we calculate the product

$$\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}} = \omega R_Z\big(\cos\theta\ (\boldsymbol{\mathcal{F}} \cdot \mathbf{n}) - \sin\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{t})\big) \tag{78}$$

in order to take an expression for the rate of dissipation in terms of the configurational force and the configurational moment at the crack tip

$$\Phi = \frac{a}{R_Z \sin\theta}(\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}) - V\cot\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{n}). \tag{79}$$

Furthermore, if we denote with

$$G_r = \frac{\Phi}{\omega} \tag{80}$$

the *rotational energy release rate*, that is, the energy flow into the crack tip per unit angle extension of the crack, then from eq. (79) we have

$$G_r = \frac{a}{R_Z \sin\theta}(\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}) - a\cot\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{n}). \tag{81}$$

Remark 6: Assuming that $\mathbf{Z}(t)$ is a C^2 function, the instantaneous radius of curvature at $\mathbf{Z}(t)$ is related to the instantaneous curvature at $\mathbf{Z}(t)$ by $a = \frac{1}{|k|}$ and

$$|k| = \frac{|\frac{d\mathbf{Z}}{dt} \times \frac{d^2\mathbf{Z}}{dt^2}|}{|\frac{d\mathbf{Z}}{dt}|^3} = \frac{|\mathbf{V} \times \mathbf{A}|}{|\mathbf{V}|^3}, \tag{82}$$

where $\mathbf{A} = d^2\mathbf{Z}/dt^2$ is the acceleration vector of the crack tip. Therefore, we can write Φ (eq. (79)) in the following alternative form

$$\Phi = \frac{V^2}{R_Z A_n \sin\theta}(\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{M}}) - V\cot\theta(\boldsymbol{\mathcal{F}} \cdot \mathbf{n}), \tag{83}$$

where A_n is the normal component of the vector $\mathbf{A}$.

Remark 7: In the case in which the crack is circular and the origin of the coordinates' system coincides with the center of the circle, the formulas (79) and (81) give

$$\Phi = \omega \cdot \boldsymbol{\mathcal{M}}, \quad G_r = \boldsymbol{\mathcal{M}} \cdot \mathbf{m} = \mathcal{M}. \tag{84}$$

We can see from eq. $(84)_2$ that the rotational energy release rate is simply the magnitude of the vector of the configurational moment at the crack tip. Analogous results to eq. $(84)_1$ have been provided by (Maugin and Trimarco 1995) as well as by (Budiansky and Rice 1973) for disclinations and cavities, respectively. Of course, someone can easily see the analogy between the relations (68), (69) and $(84)_1$, $(84)_2$, respectively.

7 CONCLUSIONS

The objective of this paper was the study of the crack propagation within an elastic medium in the context of configurational mechanics. To this end, we proposed an appropriate kinematics and we formulated the corresponding transport and divergence theorems. In the sequel, we produced a rigorous localization process which has been used to derive the local equations for both the physical and configurational fields.

A significant consequence of the localization process was the expression for the configurational force at the crack tip related to the J–integral as well as the corresponding one for the configurational moment at the crack tip which is related to the L–integral. Based on these expressions, we derived a relationship between the configurational force at the crack tip and the energy release rate, as well as a relation connecting the rotational energy release rate with the configurational moment and force at the crack tip.

In the case of a crack with non constant curvature, the rotational energy release rate depends essentially on the geometrical characteristics of the curve. Therefore, in order to apply the formula (81), the geometrical characteristics of the curve along which the crack will evolve should be a priori known. Such situations appear in delamination cracks, where the crack necessarily follows a particular curve.

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